On the Relation Between Modes in Rectangular, Elliptical, and Parabolic Waveguides and a Mode-Classifying System

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Abstract—The class of waveguides with rectangular, elliptical, and parabolic cross sections and their transition shapes has been investigated. The transition shapes have been described by hyperelliptic functions. By a numerical procedure based on the finite-element method, the cutoff wavelength and the mode patterns of 12 of the lowest-order TM modes and 14 of the lowest-order TE modes of this class have been found. On the basis of the investigation, a mode-classifying system for arbitrary waveguide cross sections is suggested.

I. Introduction

IT IS THE PURPOSE of this paper to describe the correspondence between the modes of rectangular, elliptical, and parabolic waveguides and, based on this, to suggest a system for classifying modes in waveguides of arbitrary shape. The need for such a classifying system has become of increasing interest with the numerical methods now available for investigating waveguides of arbitrary shape. It is further emphasized by the fact that the same waveguide mode has different names in the various classical waveguide types, the cross sections of which belong to separable coordinate systems. The modes of these guides are each named according to their corresponding coordinate system.

The idea used in defining the classifying system is that any mode in an arbitrary waveguide corresponds to a certain mode in a standard system, and that it should carry the name of this standard mode and be recognized by the pattern of this standard mode. As standard reference is used, the most well-known and easiest handled waveguide system, the rectangular, in which it is very easy to find the mode patterns for as many high-order modes as wanted.

Although the waveguide cross sections investigated here belong to a certain class (those which are oblong, convex, and have two lines of symmetry only, Fig. 1), the suggested classifying system is believed to be applicable to other cross sections, too. However, before all possible cross sections can be included in the system, more investigations are needed. This is especially the case for cross sections with more lines of symmetry (degenerate modes) and for complicated concave cross sections, which in certain cases may tend to split up the mode patterns into more and almost independent areas.

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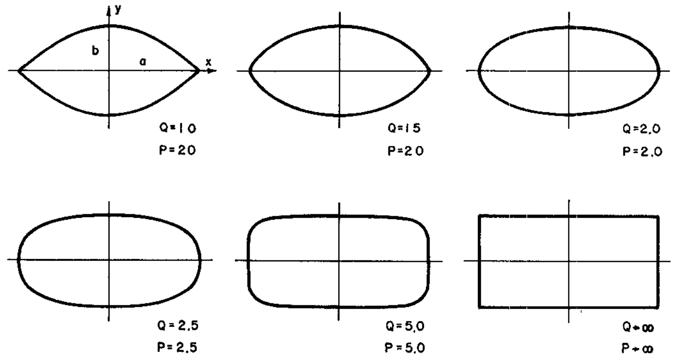


Fig. 1. Hyperelliptic waveguide cross sections investigated in the paper.

As an example of the applicability of the suggested system, Section VII shows how the system applies to a waveguide whose cross section has no lines of symmetry.

The investigation is mainly numerical and no rigorous mathematics have been included, except the results for special cases of waveguides with cross sections described by a separable coordinate system.

II. Method of Approach

One of the main ideas of the study described here is to follow the variation of the individual modes when the waveguide cross section is changed gradually in some way between the rectangular, elliptical, and parabolic shapes. In order to describe the transition shapes with few parameters, the so-called hyperelliptic functions are chosen. These functions are an extension of the superelliptic functions, which were first applied for architectural purposes, but also have been introduced in electromagnetics [1].

A superelliptic cross section is described with reference to Fig. 1 by

$$\frac{|x|^n}{a^n} + \frac{|y|^n}{b^n} = 1 \tag{1}$$

where n may take any value. By means of the three parameters a, b, and n, this curve may be changed continuously between any circle, ellipse, and rectangle. For $a=b$ and $n=2.0$, a circle is obtained; for $a \neq b$ and $n=2.0$, an ellipse is obtained; and for $a \neq b$ and $n \to \infty$, a rectangle is obtained.

In the hyperelliptic cross section, the exponents of the two terms containing x and y are allowed to be unequal:

$$\frac{|x|^P}{a^P} + \frac{|y|^Q}{b^Q} = 1. \qquad (2)$$

For $P=Q=2.0$, (2) describes an ellipse. When $P=2.0$ and $Q=1.0$, it describes the cross section of a parabolic waveguide. When Q is changed gradually from 2.0 to 1.0, the cross section is changed gradually from the elliptic to the parabolic.

The correspondence between the parameters P and Q and the waveguide cross section described by (2) is shown in Fig. 1. These waveguide cross sections have been investigated numerically by a computer program constructed by Pontoppidan [2].

In order to include the parabolic waveguide in the investigation, it has been chosen to restrict the cross sections investigated numerically to those having a ratio $b/a=0.5$ between the dimensions b and a in the symmetry planes. Twelve of the lowest-order TM modes and fourteen of the lowest-order TE modes have been found.

It is interesting to note that some of the modes in the elliptical and parabolic waveguides, which could be predicted to exist by the method of gradually changing the rectangular cross section to the elliptical and parabolic, were not found by exact theories until recently. The elliptical ones were found by Kretzschmar [3] in 1970 and the parabolic ones by Zagrodzinski [4] in 1966.

In the following, it will first be shown how the mode patterns, which are the important tools in the suggested mode-classifying system, will be displayed. Next, the exact results known from the literature about the classical waveguide types will be mentioned, and, finally, these results will be shown in connection with the new numerical results for the hyperelliptic waveguides of which they are special cases.

III. Illustrating Waveguide Modes

The usual method to illustrate the configuration of the various waveguide modes is to plot both electric and magnetic field lines in a typical cross section. In order to simplify the recognition of the various mode patterns, only one set of field lines will be used here. For TE modes the electric field line patterns will be used and for TM modes the magnetic field line patterns will be used.

A simple way of finding these field line patterns is to plot the contour lines of the generating wave functions, as shown, e.g., by Pontoppidan [2]. The generating wave function is the longitudinal H field of the guide for TE modes and the longitudinal E field of the guide for TM modes. The field patterns shown in this paper are plotted by a computer using a contour-plotting subroutine [5] for the generating functions. All mode patterns are normalized to have a maximum field strength of unity.

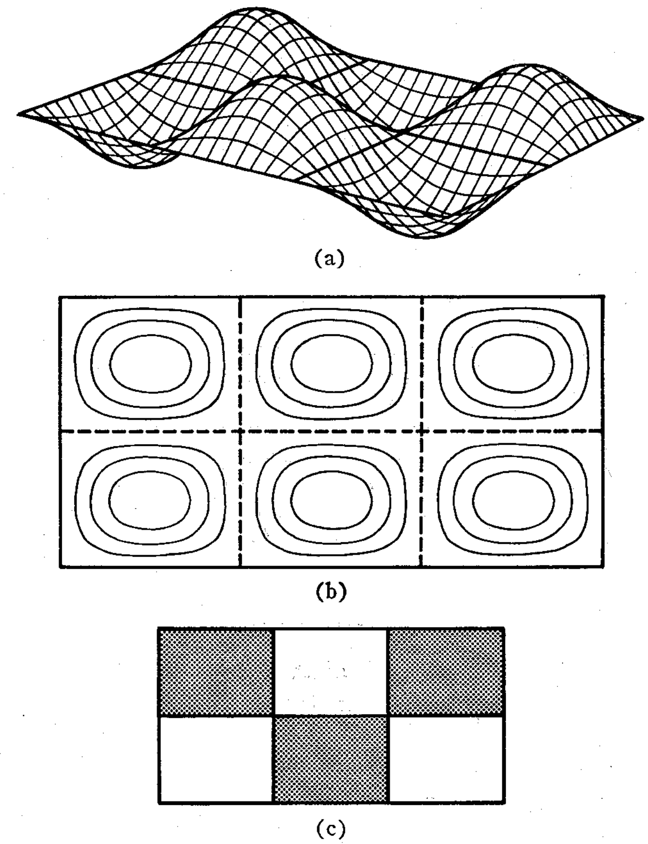


Fig. 2. Illustration of a TM mode (as an example, the TM_{32} mode of a rectangular waveguide is shown). (a) Perspective view of the generating function equal to the longitudinal E field. (b) Magnetic field line pattern equal to the contour lines of the generating function. (c) Sketch of the mode.

To illustrate the principle of this procedure, perspective drawings of the generating function and the corresponding system of contour curves, which represent the field pattern for a TM and a TE mode, respectively, are shown in Figs. 2 and 3. The dashed lines are the contour lines corresponding to a field strength of zero. At the bottom of each of the figures a simple and less-informative sketch of the modes is shown, where hatched areas correspond to a positive value of the generating functions and blank areas to a negative one. This signal-flag way of designating waveguide modes appears to be quite practical in many cases, as illustrated in what follows.

IV. Elliptical Waveguides

Waveguides of elliptical cross section were first treated in 1936 in the classical paper by Chu [6]. Several investigations have followed this; however, not until the recent investigations made by Kretzschmar [3], all of the lowest-order modes of these waveguides have been found. This course of development is due to the fact that the Mathieu functions describing the field in elliptical waveguides are difficult to compute numerically because of a rather slow convergence of the series representing the functions. Kretzschmar has, by using a computer program based on a Bessel-functions product

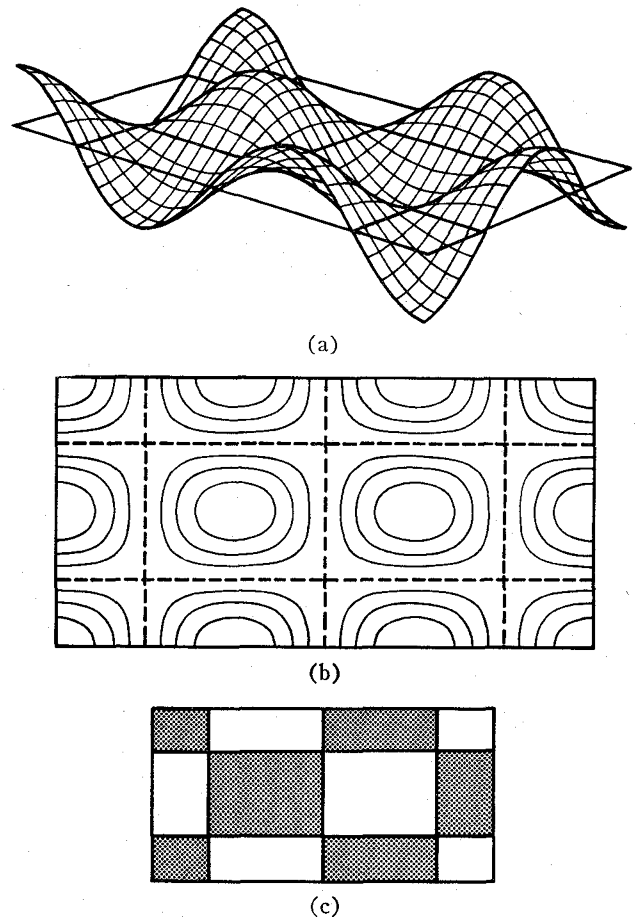


Fig. 3. Illustration of a TE mode (as an example, the TE_{32} mode of a rectangular waveguide is shown). (a) Perspective view of the generating function equal to the longitudinal H field. (b) Electric field line pattern equal to the contour lines of the generating function. (c) Sketch of the mode.

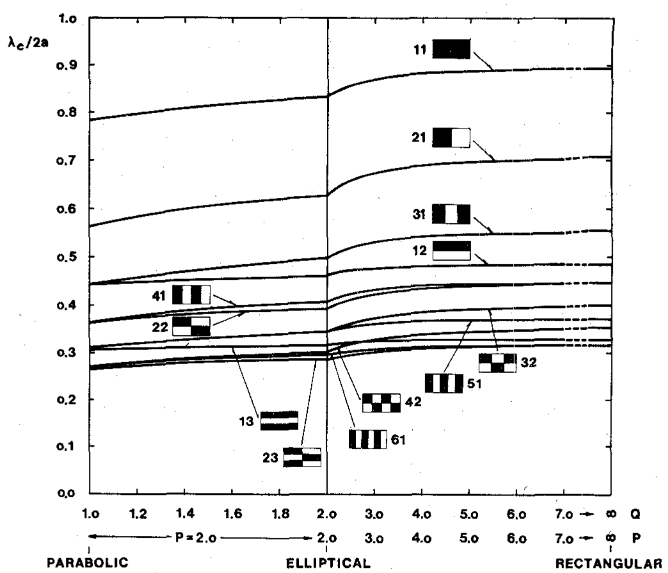


Fig. 4. Normalized cutoff wavelength of TM modes in waveguides with hyperelliptic cross section; $b/a=0.5$.

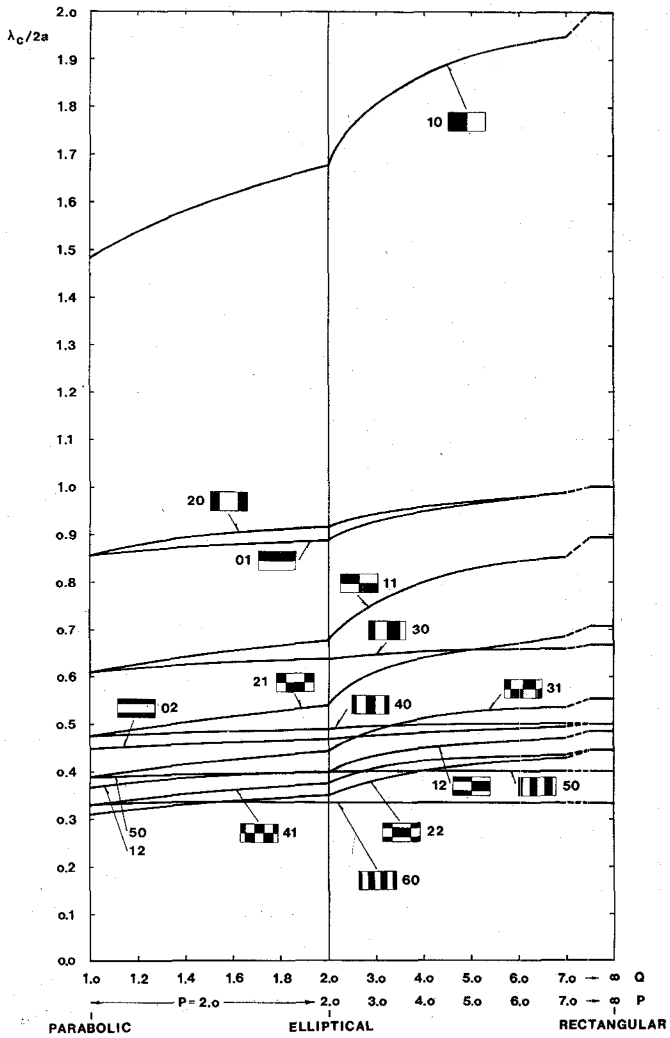


Fig. 5. Normalized cutoff wavelength of TE modes in waveguides with hyperelliptic cross section; $b/a=0.5$.

series for the Mathieu functions, found the 19 lowest-order modes (11 TE modes and 8 TM modes) of an elliptical waveguide. However, he does not show the mode patterns, and only a few of these have been published previously.

Of the previous papers on elliptical waveguides, reference should be made to those by Piefke [7] and by Krank [8], from which the mode names of the elliptical guide in Figs. 6 and 7 and the mode patterns of the elliptical guide in Fig. 9 were found.

V. Parabolic Waveguides

The first investigation of parabolic waveguides was published in 1942 by Spence and Wells [9], and several papers on the subject have appeared since then. However, not until 1966 (published 1968) did Zagrodzinski [4] find the parameter values yielding all the lowest-order modes of the parabolic waveguides. In earlier work, only the modes, which correspond to the zero order of the parabolic cylinder functions, were found, whereas the more complicated modes, which correspond to non-zero orders, were mentioned, but not investigated. Zagrodzinski found by graphical means, using Miller's

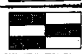
Mode pattern	Rectangular		Elliptical		Parabolic	
	Name	$\lambda_c/2a$	Name	$\lambda_c/2a$	Name	$\lambda_c/2a$
	TM_{11}	0.894	TM_{c01}	0.831	TM_{00}	0.785
	TM_{21}	0.707	TM_{c11}	0.627	TM_{11}	0.563
	TM_{31}	0.555	TM_{c21}	0.496	$TM_{20} - TM_{02}$	0.442
	TM_{12}	0.485	TM_{s11}	0.458	$TM_{20} + TM_{02}$	0.442
	TM_{41}	0.447	TM_{c31}	0.407	$TM_{31} - TM_{13}$	0.363
	TM_{22}	0.447	TM_{s21}	0.394	$TM_{31} + TM_{13}$	0.363
	TM_{32}	0.400	TM_{s31}	0.343	$TM_{40} + TM_{04}$	0.312
	TM_{51}	0.371	TM_{c41}	0.344	$TM_{40} - TM_{04}$	0.312
	TM_{42}	0.354	TM_{s41}	0.302	$TM_{51} + TM_{15}$	0.270
	TM_{13}	0.329	TM_{c02}	0.316	TM_{22}	0.306
	TM_{61}	0.316	TM_{c51}	0.297	$TM_{51} - TM_{15}$	0.270
	TM_{23}	0.316	TM_{c12}	0.284	TM_{33}	0.266

Fig. 6. Table of corresponding mode designations for TM modes in rectangular, elliptical, and parabolic waveguides. The numerical values of the normalized cutoff wavelength are valid for $b/a=0.5$.

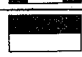
Mode pattern	Rectangular		Elliptical		Parabolic	
	Name	$\lambda_c/2a$	Name	$\lambda_c/2a$	Name	$\lambda_c/2a$
	TE_{10}	2.000	TE_{c11}	1.676	TE_{11}	1.481
	TE_{20}	1.000	TE_{c21}	0.919	$TE_{20} - TE_{02}$	0.856
	TE_{01}	1.000	TE_{s11}	0.888	$TE_{20} + TE_{02}$	0.856
	TE_{11}	0.894	TE_{s21}	0.677	$TE_{31} + TE_{13}$	0.609
	TE_{30}	0.667	TE_{c31}	0.637	$TE_{31} - TE_{13}$	0.609
	TE_{21}	0.707	TE_{s31}	0.538	$TE_{40} + TE_{04}$	0.474
	TE_{40}	0.500	TE_{c41}	0.489	$TE_{40} - TE_{04}$	0.474
	TE_{02}	0.500	TE_{c01}	0.465	TE_{22}	0.449
	TE_{31}	0.555	TE_{s41}	0.443	$TE_{51} + TE_{15}$	0.389
	TE_{12}	0.485	TE_{c12}	0.397	TE_{33}	0.367
	TE_{50}	0.400	TE_{c51}	0.399	$TE_{51} - TE_{15}$	0.389
	TE_{41}	0.447	TE_{s51}	0.375	$TE_{60} + TE_{06}$	0.329
	TE_{22}	0.447	TE_{c22}	0.347	$TE_{42} - TE_{24}$	0.310
	TE_{60}	0.333	TE_{c61}	0.337	$TE_{60} - TE_{06}$	0.329

Fig. 7. Table of corresponding mode designations for TE modes in rectangular, elliptical, and parabolic waveguides. The numerical values of the normalized cutoff wavelength are valid for $b/a=0.5$.

tables [10] of parabolic cylinder functions, the cutoff frequencies of the 143 lowest-order modes in a parabolic waveguide. The present author has supplemented Zagrodzinski's results by making a program for computing the odd and even parabolic cylinder functions [11], and used this in connection with the previously mentioned contour-plotting procedure to find the mode patterns of the parabolic waveguide.

VI. Mode System by Numerical Solution

The cutoff wavelength of the transition shapes between the rectangular, elliptical, and parabolic waveguides described by the hyperelliptic functions have been found approximately by an iterative finite-element program constructed by Pontoppidan [2]. The results are shown in Fig. 4 for the TM modes and in Fig. 5 for the TE modes. The cutoff wavelengths are normalized with the major axis $2a$ of the cross sections. The mode curves have been marked with the mode indices of the corresponding rectangular waveguide mode and with the signal-flag sketches of the same modes.

Further information related to the curves of Figs. 4 and 5 are given in the table of Fig. 6 for the TM modes and in the table of Fig. 7 for the TE modes. In the tables are listed the same signal-flag designation as was used for the curves, and, furthermore, the normal mode classifications of the classical waveguide cross sections have been given.

The corresponding mode designations of the rectangular and the elliptical waveguides are valid for all values of the ratio b/a. The names of the parabolic waveguide modes are only defined for $b/a=0.5$. The numerical values in the tables of the normalized cutoff wavelengths are valid for $b/a=0.5$.

In order to show how much the mode patterns change for the relatively similar cross sections treated here, examples are shown in Fig. 8 for the corresponding patterns of the TM_{41} mode and the TE_{21} mode of the three classical waveguide types. These modes are easy to recognize. In Fig. 9 examples are shown, with the aid of the signal-flag technique, for the corresponding patterns of the TM_{13} and TE_{13} modes of the three waveguide types. These modes are more difficult to recognize; however, with a little experience it should be possible, and there seems to be an advantage in the signal-flag technique when more complicated modes are to be identified.

VII. Mode-Classifying System

In the preceding section it was shown how there is a unique correspondence between the modes of parabolic, elliptical, and rectangular waveguides. The mode-classification system to be suggested here is based on the anticipation that a similar correspondence exists between modes of waveguides with other cross sections than those investigated here.

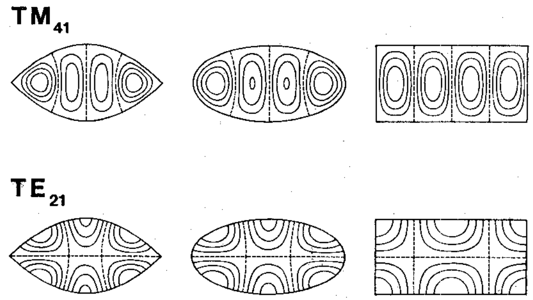


Fig. 8. Example of corresponding modes in parabolic, elliptical, and rectangular waveguides.

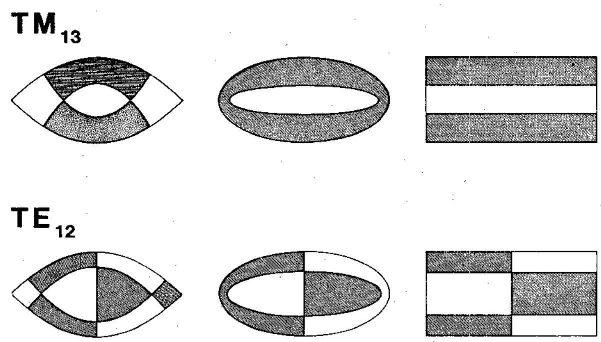


Fig. 9. Example of corresponding modes in parabolic, elliptical, and rectangular waveguides.

In accordance with this, it is suggested to classify modes in waveguides by the name and pattern of the corresponding mode in a rectangular waveguide. By corresponding mode it is meant that mode of a rectangular waveguide into which the actual mode transfers when the actual waveguide cross section gradually is changed to a rectangular one in a simple way.

For waveguides belonging to the class defined in the Introduction (Fig. 1), there are no problems in finding a simple way of transition, as the axes of the symmetry and the way of being oblong should be the same for the cross section investigated and for the standard rectangular waveguide.

For other cross sections there might be some doubt as to how the orientation of the waveguide investigated should be in relation to the standard guide. However, this ambiguity can be solved by finding the way into which the lowest-order TE mode, TE_{10}, or the second lowest-order TM mode, TM_{21}, divides the cross section. The contour lines of these modes, which correspond to the zero field value, should be oriented in the same way as the corresponding lines of the standard pattern. As an example of this procedure, Fig. 10 shows the six lowest-order TM modes of an oblong waveguide cross section without any lines of symmetry. The cross section, which was investigated by the program of Pontoppidan [2], consists of a semicircle and a right isosceles triangle. In the figure, a corresponding set of standard

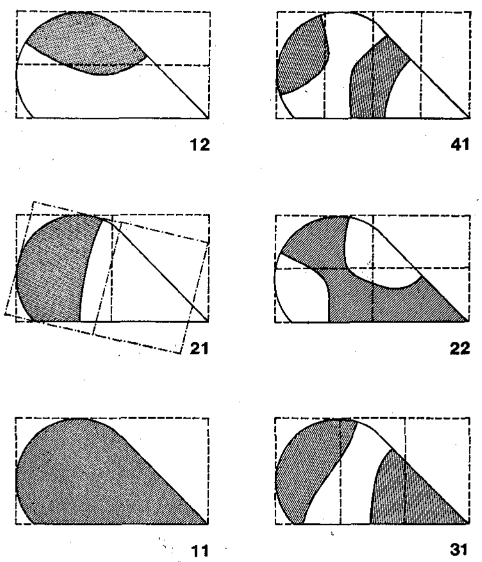


Fig. 10. Example of classification of TM modes in a waveguide with unsymmetrical cross section. A corresponding set of rectangular guide patterns is shown with dashed lines.

rectangular waveguides is shown. The mode names have easily been found as indicated. The standard set shown is not the only one that could be used (another example is shown dotted at the 21 mode). However, the mode names are independent of the choice of standard set, if it is chosen according to the rules given above.

When a waveguide cross section has more than two lines of symmetry like the equilateral triangle and the other equilateral polygons of which the circle is a limiting case, several degenerate modes may exist, and it is not possible to uniquely decide which one is the TE_{10} or TM_{21} mode. Furthermore, new mode patterns, which do not exist in the rectangular standard system, may arise by a linear combination of the degenerate modes. The present system of classification does not consider modes of this kind.

VIII. Conclusion

The correspondence between modes in rectangular, elliptical, and parabolic waveguides have been investigated numerically. Based on the results, a mode-classifying system for arbitrary waveguides has been suggested. The system has, by examples, been shown to be applicable for lower-order modes in waveguides with not too complicated cross section. The patterns arising from a linear combination of degenerate modes in special cross sections have not been included in the system.

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